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Ski Rental Problem with Dynamic Pricing

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Abstract. We extend the classical ski rental problem, so that the rental price may evolve with time. We consider several models which differ by the knowledge given to the algorithm: the algorithm may have full, partial or no knowledge about the duration of the game. We construct algorithm which achieve up to constant factor optimal competitive ratios.

1 Introduction

In the classic ski rental problem, a skier may rent skis for p dollars per day or buy them for $s \cdot p$ dollars, where s is an integer greater than one. At the end of any day, the skier may break his legs along with the skis, or in some other way irrevocably finish skiing. The goal is to develop an online strategy minimizing the cost spent on skiing, where the cost is compared to the cost of an optimal offline strategy for the same input. The worst-case ratio between these two amounts is called competitive ratio.

The well-known result [8, 9] states that the best online strategy is to rent skis for $s-1$ day and then buy them on day s. It is straightforward that such a strategy is $(2 - 1/s)$-competitive.

In this paper, we extend this model, so that the rental price p may evolve with time. Therefore, the instance of the problem is not only the duration of the game in days, but also a sequence of prices at consecutive days. However, if we do not impose any constraints on the way the price changes, no algorithm may achieve a finite competitive ratio (even if the game is guaranteed to last infinitely). To see this, fix any k and assume that at day one $p = 1$. If a skier buys skis at the first day, then the future prices are equal to $1/k$, otherwise they are equal to k. Obviously, the optimal solution in these cases is to buy at the second day or at the first one, respectively, and the competitive ratio is $\Omega(k)$.

Therefore, we assume that the rate of price changes is bounded, i.e., the price for renting skis at any day is at least 1 and the prices in two consecutive days differ at most by 1. Apparently, in such a scenario, the competitive ratios are quite high. Therefore, we investigate variants, in which algorithm has full or partial knowledge about the duration of the game. In particular, we consider a hybrid scenario in which the duration of the game is a random variable, corresponding to the following natural model: each day skier quits with probability $1/\lambda$ and continues skiing with probability $1 - 1/\lambda$.

In this paper, we consider a continuous variant of the problem. Namely, the prices are given as a continuous curve satisfying Lipschitz condition and skier is renting skis up to time t in which he buys skis (t is possibly a non-integer). This almost does not change the problem: the algorithm has to be a bit more careful in the initial $[0, 1]$ period of the game. On the other hand, in the continuous model the calculations are easier and we show that our algorithms work can be modified to work in the discrete model without increasing their competitive ratios.

1.1 Problem formulation and our results

We assume that the input instance is a pair: an infinite *curve of prices* and a *game duration*. The former is a continuous function p_t satisfying Lipschitz condition, i.e. for any two times t_0 and t_1, it holds that $|p_{t_1} - p_{t_0}| \leq |t_1 - t_0|$. The latter is a real positive number γ.

In online analysis, the input is revealed to the algorithm element by element: in the discrete counterpart of the problem, the algorithm would be given prices in consecutive days. We would like to formulate algorithms in a similar manner, i.e. say e.g. that "algorithm waits for the price p to reach a given threshold". While we use such phrases in the paper, we note that they can be justified, without having the

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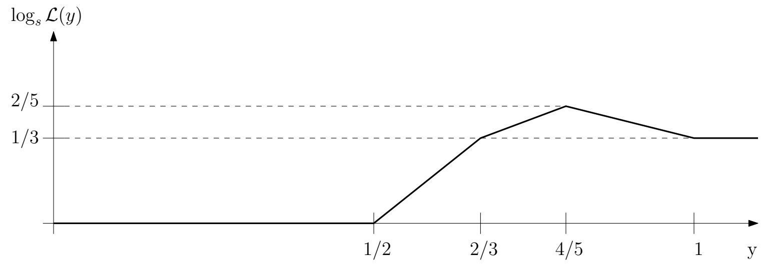

y	$(-\infty, 1/2)$	$[1/2, 2/3)$	$[2/3, 4/5)$	$[4/5, 1)$	$[1, \infty)$
$\mathcal{L}(y)$	1	s^{2y-1}	$s^{y/2}$	$s^{(2-y)/3}$	$s^{1/3}$

Fig. 1. Function $\mathcal{L}(y)$ used in the description of competitive ratios

algorithm to process uncountable number of input elements. For example, the algorithm may choose a *probing frequency* ϵ, read the curve of prices at times $0, \epsilon, 2\epsilon, 3\epsilon \ldots$ and make decision whether to buy skis also at these times. The last value presented to the algorithm is then $\lfloor \frac{\gamma}{\epsilon} \rfloor \cdot \epsilon$. As the price curve is continuous, for a small value of ϵ this read model gives the algorithms essentially the same power as the model in which the algorithm may buy skis at any time.

Fix any input curve p_t and the game duration γ. Let $\mathcal{F}(T) = \int_{t=0}^{T} p_t \, dt$. Let BUY$(T)$ be an algorithm, which buys at time T. In particular, let BUY(∞) be an algorithm, which always rents. Then its cost is equal to

$$C_{\text{BUY(T)}} = \begin{cases} \mathcal{F}(T) + s \cdot p_T & \text{if } T \leq \gamma \\ \mathcal{F}(T) & \text{if } T > \gamma \end{cases} .$$

The cost of the optimal offline solution, OPT, is equal to $C_{\text{OPT}} = \min_{0 \leq T \leq \infty} \{C_{\text{BUY(T)}}\}$

For any deterministic algorithm ALG, we say that it is $\mathcal{R}$-competitive if for any input it holds that $C_{\text{ALG}}/C_{\text{OPT}} \leq R$. If the algorithm is randomized, we replace C_{ALG} by its expected value. We usually view the process as a game between the algorithm and the adversary. For randomized algorithms we assume oblivious adversaries, which does not see random bits used by the algorithm.

We consider several scenarios, which differ in the knowledge given to the algorithm.

Scenario A: Unknown game end. This is the most straightforward approach, in which the adversary dictates both prices curve p_t, duration γ of the game, and the algorithm learns the game end at time γ. For this scenario, in Section 2, we give a deterministic $\mathcal{O}(\sqrt{s})$-competitive algorithm. This ratio is asymptotically optimal, as we present an up to constant factor matching lower bound, which holds even for randomized algorithms.

Scenario B: Known game end. In this scenario, the adversary creates a price curve and fixes duration γ, but the algorithm learns γ at the very beginning and can adjust its behavior accordingly. For describing our results, we define a function $\mathcal{L}$ as in Figure 1. For any given γ, in Section 3, we construct an algorithm PROT$_\gamma$ which is $\mathcal{O}(\mathcal{L}(\log_s \gamma))$-competitive for Scenario B. We provide an asymptotically matching lower bound for any randomized algorithm in Section 5.

Scenario C: Stochastic game end. In this scenario, the adversary creates a prices curve and a parameter λ, which is revealed to the algorithm at the very beginning. Then the duration of the game γ is chosen randomly according to the exponential distribution with parameter λ. The algorithm learns the game end at time γ. As mentioned above, this is a continuous counterpart of the following natural model: the probability of breaking leg is $1/\lambda$ each day, i.e. the expected duration is λ. In Section 4, we show that in this scenario the expected value of the competitive ratio of PROT$_\lambda$ is $\mathcal{O}(\mathcal{L}(\log_s \lambda) \cdot \log \min\{\lambda, s\})$.

Finally, in Section 6, we show that our algorithms for continuous model can be made to work in the discrete model without increasing their competitive ratios.

1.2 Related Work

For the classic ski rental problem, the competitive ratio of the best deterministic online algorithm is $2 - 1/s$ [8, 9]. If we allow randomization, then this ratio can be reduced to $\mathrm{e}/(\mathrm{e}-1) \approx 1.58$ [7]. Similar, optimal ratios (2 for the deterministic case and $\mathrm{e}/(\mathrm{e}-1)$ for the randomized case) were shown for related rent-or-buy problems like Bahncard problem [4, 6] or TCP acknowledgement problem [1, 6].

Ski rental problem was also analyzed in average-case: Fujiwara and Iwama considered the case when the duration of the game is determined by a stochastic process and the goal is to minimize the expected value of the competitive ratio [5]. In particular, if the duration is an exponentially distributed (with parameter λ) random variable, they proved that the optimal strategies are to rent forever if $\lambda \leq s$, or buy skis after renting them for s^2/λ days if $\lambda > s$.

To our best knowledge, ski rental problem was not analyzed in a model in which prices may evolve with time. This problem is loosely related to various versions of currency trading (see e.g. [3, 2] and the references therein). Due to different focus and assumptions, algorithmic ideas developed there do not seem to apply to our problem.

2 Unknown Game End

Let algorithm Thresh be the following deterministic algorithm for Scenario A. In period $[0, \sqrt{s})$, Thresh always rents. Then it buys skis at time $k \geq \sqrt{s}$ if $\mathcal{F}(k) \geq s$.

Theorem 1. Thresh *is $\mathcal{O}(\sqrt{s})$-competitive for Scenario A.*

Proof. Let γ be the duration of the game. Let k be the time of skis purchase or $k = \gamma$ if Thresh does not buy skis.

Assume that Thresh buys skis, i.e. $C_{\text{THRESH}} = \mathcal{F}(k) + s \cdot p_k$, and we relate p_k to $\mathcal{F}(k)$. If $p_k \leq 2 \cdot \sqrt{s}$, then $\mathcal{F}(k) \geq s \geq \sqrt{s} \cdot p_k/2$. If $p_k > 2\sqrt{s}$, then in period $[k - \sqrt{s}, k]$ the price is at least $p_k - \sqrt{s} \geq p_k/2$. Thus, in this case, $\mathcal{F}(k) \geq \sqrt{s} \cdot p_k/2$ as well.

In total, it means that $C_{\text{THRESH}} = \mathcal{O}(\sqrt{s}) \cdot \mathcal{F}(k)$. Obviously, the same bound holds if Thresh does not buy skis for the whole game ($k = \gamma$). Therefore, it suffices to show that $\mathcal{F}(k) \leq 2 \cdot C_{\text{OPT}}$.

If Opt rents forever or buys skis at time k or later, then clearly $C_{\text{OPT}} \geq \mathcal{F}(k)$. Otherwise, Opt buys skis at time $\ell < k$, paying at least $s \cdot p_\ell$. We consider two cases.

1. $k > \sqrt{s}$. By the construction of Thresh, k is the first time for which $\mathcal{F}(\cdot)$ reaches s. Thus, $\mathcal{F}(k) = s \leq C_{\text{OPT}}$.
2. $k \leq \sqrt{s}$. As any two prices in period $[0, k]$ can differ at most by k, we obtain

$$\mathcal{F}(k) = \int_0^k p_t \, dt \leq \int_0^k (p_\ell + k) \, dt \leq k \cdot p_\ell + k^2 \leq 2 \cdot s \cdot p_\ell \leq 2 \cdot C_{\text{OPT}} .$$

□

We show that Thresh achieves asymptotically optimal competitive ratio, by proving that the ratio holds even for randomized algorithms.

Theorem 2. *The competitive ratio of any randomized algorithm against an oblivious adversary for Scenario A is $\Omega(\sqrt{s})$.*

Proof. In our lower bounds, we employ Yao min-max principle [10], i.e. we construct a probability distribution π over inputs and show that on the expectation no *deterministic* algorithm can achieve better competitive ratio.

Our approach uses only prices from the range $[1, \sqrt{s}+1]$. The price curve is fixed, $p_t = 1 + \min\{t, \sqrt{s}\}$. The game duration γ is chosen randomly. With probability $1/2$, $\gamma = \infty$. The remaining probability is distributed uniformly in the range $[0, \sqrt{s}]$, i.e. the density function is equal to $f(x) = 1/(2 \cdot \sqrt{s})$ in this range and 0 outside it.

We look at a deterministic algorithm $\textsc{Det}$, which is the best algorithm against this distribution. Let $C_{\textsc{Det}}(x)$ be the cost of $\textsc{Det}$, provided it is run on a sequence of length x. We use analogous definition for $\textsc{Opt}$. By the Yao min-max principle, it follows that the competitive ratio $\mathcal{R}$ of any randomized algorithm is at least the expected performance of $\textsc{Det}$, i.e.

$$\mathcal{R} \geq \mathbf{E}_\pi \left[\frac{C_{\textsc{Det}}}{C_{\textsc{Opt}}}\right] = \frac{1}{2} \cdot \frac{C_{\textsc{Det}}(\infty)}{C_{\textsc{Opt}}(\infty)} + \int_0^{\sqrt{s}} \frac{C_{\textsc{Det}}(\alpha)}{C_{\textsc{Opt}}(\alpha)} f(\alpha)\, d\alpha \tag{1}$$

We consider two cases.

1. $\textsc{Det}$ chooses to buy skis after time $\sqrt{s}/2$ or always rents. Then $C_{\textsc{Det}}(\infty) = \Omega(s \cdot \sqrt{s})$. On the other hand $C_{\textsc{Opt}}(\infty) = s$, as $\textsc{Opt}$ may buy skis at time 0. Thus, $\mathcal{R} = \Omega(\sqrt{s})$ in this case.
2. $\textsc{Det}$ chooses to buy skis at time $r \leq \sqrt{s}/2$. We note that $\textsc{Opt}(\alpha) = \mathcal{O}(\alpha^2)$. In this case, we get

$$\begin{aligned} \mathcal{R} &\geq \int_0^{\sqrt{s}} \frac{C_{\textsc{Det}}(\alpha)}{C_{\textsc{Opt}}(\alpha)} f(\alpha)\, d\alpha \\ &= \Omega(1) \cdot \frac{1}{2 \cdot \sqrt{s}} \cdot \int_r^{\sqrt{s}} \frac{s \cdot r}{\alpha^2}\, d\alpha \\ &= \Omega(1) \cdot \frac{1}{2 \cdot \sqrt{s}} \cdot s \cdot r \cdot \left(\frac{1}{r} - \frac{1}{\sqrt{s}}\right) \\ &= \Omega(\sqrt{s}) \end{aligned}$$

Thus, in any case, the competitive ratio is at least $\Omega(\sqrt{s})$, and the theorem follows. □

3 Algorithm PROT

In this section, we construct an algorithm $\textsc{Prot}_\lambda$, which we further analyze in Scenarios B and C. λ is a parameter of the algorithm; the algorithm performs well if $\lambda = \gamma$. This can be guaranteed in Scenario B; in Scenario C, $\gamma \sim \text{Exp}(\lambda)$, hence our bounds are slightly weaker there.

We compute the bounds on the competitive ratio of $\textsc{Prot}_\lambda$ provided it is run on a sequence of length γ. We denote the competitive ratio in such a setting by $\mathcal{R}_{\textsc{Prot}}(\lambda, \gamma)$. Then the actual competitive ratio of $\textsc{Prot}_\lambda$ in Scenario B is given by $\mathcal{R}_{\textsc{Prot}}(\lambda, \lambda)$. The expected value of the competitive ratio in Scenario C is given by $\mathbf{E}_\gamma[\mathcal{R}_{\textsc{Prot}}(\lambda, \gamma)]$.

3.1 Algorithm $\textbf{PROT}_\lambda$

The behavior of algorithm $\textsc{Prot}_\lambda$ depends on the parameter λ. We think of λ as the algorithm's best known estimate of γ. As in the usual ski rental problem, we want to amortize the cost of skis purchase against the cost of their rental. For example, if λ is small, it makes little sense to buy skis.

We give an informal intuition for Scenario B and algorithm $\textsc{Prot}_s$ run on an input of length s. The algorithm has to protect itself against two different types of inputs. An input may contain a time with a low price (e.g. 1) and $\textsc{Opt}$ buys for this price. For these inputs, $\textsc{Prot}_s$ chooses a threshold A and buys skis when the price goes below A. On the other hand, prices may be growing for the whole input. In this case the solution for $\textsc{Opt}$ is to buy at time 0. $\textsc{Prot}_s$ tries to mimic this behavior by the second threshold B; it buys skis while the price goes above B.

Although it does not change the behavior of the algorithm in Scenario B, we assume that the algorithm does not buy skis (even if thresholds are reached) in period $[0, 1)$. The reason for this stems from Scenario C: in this case the adversary could trigger the skis purchase of $\textsc{Prot}$ at time 0, i.e. $C_{\textsc{Prot}} = s \cdot p_0$. It is possible (although rather unlikely) that the sequence ends at time ϵ, where ϵ is very small. In this case, $C_{\textsc{Opt}} \approx \epsilon \cdot p_0$, and the competitive ratio would be arbitrarily high.

Taking the intuitions above into account, we formally define $\textsc{Prot}_\lambda$ as follows.

– $\lambda < s^{2/3}$. In this case PROT_λ never buys skis.
– $s^{2/3} \leq \lambda < s^{4/5}$. PROT_λ does not buy skis in period $[0, 1)$. Later, it uses a threshold $A = s^{(3/2)y-1}$, where $y = \log_s \lambda$. If at any time price is at or below A, then PROT_λ buys skis. This also includes the case $p_1 \leq A$: then PROT_λ buys skis immediately at time 1.
– $\lambda \geq s^{4/5}$. Again, PROT_λ does not buy skis in period $[0, 1)$. Later, it uses two thresholds A and B, which are defined as follows.

$\lambda = s^y$	$[s^{4/5}, s)$	$[s, \infty)$
A	$s^{(2y-1)/3}$	$s^{1/3}$
B	$s^{(y+1)/3}$	$s^{2/3}$

If at any time price falls outside range (A, B), then PROT_λ buys at this price. This also includes a case, when p_1 is outside this range: then PROT_λ buys skis at time 1. Otherwise, if the price remains in range (A, B) for the period $[1, s]$, then PROT_λ buys at time s.

Lemma 1. *If the game ends at time $\gamma < 1$, then $\mathcal{R}_{\text{PROT}}(\lambda, \gamma) = \mathcal{O}(1)$.*

Proof. The price in the period $[0, \gamma]$ remains between $p_0 - 1$ and $p_0 + 1$. Thus, $C_{\text{PROT}} \leq \gamma \cdot (p_0 + 1) = \mathcal{O}(\gamma \cdot p_0)$. If OPT always rents, it pays the same amount as PROT_λ. Otherwise, it pays at least $s \cdot (p_0 - 1) = \Omega(\gamma \cdot p_0)$. □

Therefore in the remaining part of this section, we assume that $\gamma \geq 1$. For making our arguments concise, throughout this section, we consider a restricted version of OPT which is not allowed to buy in period $[0, 1)$. Such a restriction increases its cost at most by a constant factor, and can be neglected as we are interested in the asymptotic performance.

3.2 Algorithm Analysis: Case $\lambda < s^{4/5}$

We start with a simple observation.

Lemma 2. *Let γ be the duration of the game and $p_{\min} = \min_{0 \leq t \leq \gamma} p_t$. If PROT_λ never buys skis, then*

$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} \leq 1 + \frac{\gamma^2}{s \cdot p_{\min}} + \frac{\gamma}{s} \ .$$

Proof. Obviously, $C_{\text{PROT}} = \mathcal{F}(\gamma) \leq \gamma \cdot (\gamma + p_{\min})$ and $C_{\text{OPT}} \geq \min\{\mathcal{F}(\gamma), s \cdot p_{\min}\}$. We obtain

$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} \leq \frac{\mathcal{F}(\gamma)}{\min\{\mathcal{F}(\gamma), s \cdot p_{\min}\}} \leq 1 + \frac{\mathcal{F}(\gamma)}{s \cdot p_{\min}} \leq 1 + \frac{\gamma^2}{s \cdot p_{\min}} + \frac{\gamma}{s} \ .$$

□

We use this lemma for the case $\lambda < s^{2/3}$, taking a trivial bound $p_{\min} \geq 1$, and immediately obtaining the following theorem.

Theorem 3. *For $y < 2/3$, it holds that $\mathcal{R}_{\text{PROT}}(s^y, \gamma) = \mathcal{O}(1 + \gamma^2/s)$.*

Theorem 4. *For $y \in [2/3, 4/5)$, it holds that $\mathcal{R}_{\text{PROT}}(s^y, \gamma) = \mathcal{O}(s^{y/2} + s^{(3/2)y}/\gamma + \gamma/s + \gamma^2/s^{(3/2)y})$.*

Proof. Recall that $A = s^{(3/2) \cdot y - 1}$ is the threshold used by PROT. We consider three cases concerning the shape of the input curve.

1. The price in period $[1, \infty)$ is always above A. As PROT_λ never buys in this case, by Lemma 2, it holds that
$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} \leq 1 + \frac{\gamma^2}{s \cdot A} + \frac{\gamma}{s} \ . \quad (2)$$

2. $p_1 \leq A$. Then $C_{\text{PROT}} = s \cdot A$. On the other hand, OPT either buys skis paying at least s or it rents them for γ steps paying γ. In effect,

$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} = \mathcal{O}\left(\frac{s \cdot A}{\min\{s, \gamma\}}\right) = \mathcal{O}\left(A + \frac{s \cdot A}{\gamma}\right) \quad . \tag{3}$$

3. In period $[1, \infty)$, the price first drops to A at time $k > 1$.
If $\gamma < k$, then PROT_λ always rents and case 1 applies. Otherwise $\gamma \geq k$ and $C_{\text{PROT}} \leq \mathcal{F}(k) + s \cdot A$. OPT has essentially three choices. Either it buys in period $[1, k)$ paying at least $s \cdot A$, or it buys in period $[k, \gamma)$ paying at least $\mathcal{F}(k) + s$ or it rents forever paying at least $\max\{\mathcal{F}(k), \gamma\} \geq (\mathcal{F}(k) + \gamma)/2$. Altogether, we obtain $C_{\text{OPT}} = \min\{s \cdot A, \mathcal{F}(k) + s, (\mathcal{F}(k) + \gamma)/2\}$. Thus,

$$\begin{aligned} \frac{C_{\text{PROT}}}{C_{\text{OPT}}} &= \frac{\mathcal{F}(k) + s \cdot A}{\min\{s \cdot A, \mathcal{F}(k) + s, (\mathcal{F}(k) + \gamma)/2\}} \\ &\leq \frac{\mathcal{F}(k) + s \cdot A}{s \cdot A} + \frac{\mathcal{F}(k) + s \cdot A}{\mathcal{F}(k) + s} + \frac{2 \cdot (\mathcal{F}(k) + s \cdot A)}{\mathcal{F}(k) + \gamma} \\ &= \mathcal{O}\left(1 + \frac{\mathcal{F}(k)}{s \cdot A} + A + \frac{s \cdot A}{\gamma}\right) \\ &= \mathcal{O}\left(\frac{\gamma^2}{s \cdot A} + \frac{\gamma}{s} + A + \frac{s \cdot A}{\gamma}\right) \quad , \end{aligned} \tag{4}$$

where the last relation follows by $\mathcal{F}(k) \leq (k + A) \cdot k \leq \gamma \cdot A + \gamma^2$.

By combining (2), (3), and (4), we get that $\mathcal{R}_{\text{PROT}}(s^y, \gamma) = \mathcal{O}(A + s \cdot A/\gamma + \gamma/s + \gamma^2/(s \cdot A))$, which implies the theorem. ◻

3.3 Algorithm Analysis: Case $\lambda \geq s^{4/5}$

Before we prove the competitiveness of PROT_λ, we classify the input curve into three types.

1. An input is of type (a, b)-middle if all the prices in period $[1, s]$ are within range (a, b).
2. An input is of type (a, b)-low with parameter $k < s$ if the prices in period $[1, k)$ are within range (a, b) and $p_k \leq a$.
3. An input is of type (a, b)-high with parameter $k < s$ if the prices in period $[1, k)$ are within range (a, b) and $p_k \geq b$.

Note that for any a and b, each input is either (a, b)-low, (a, b)-middle, or (a, b)-high. Moreover, by the continuity of the prices curve, if an input is (a, b)-low with parameter k, either $k = 1$ and $p_1 \leq a$ or $k \geq 1$ and $p_k = a$. A similar relation holds for (a, b)-high sequences.

We present three lemmas describing the algorithm behavior on the corresponding three types of input sequences.

Lemma 3. *Let $\gamma \geq s^{4/5}$ and let $A < B$ be the thresholds used by PROT_λ. On a (A, B)-middle input, for $\gamma \geq 1$, $\mathcal{R}_{\text{PROT}}(\lambda, \gamma) = \mathcal{O}(B/A)$.*

Proof. If we set all the prices to A, then the cost of PROT decreases at most by $\mathcal{O}(B/A)$, whereas the cost of OPT does not increase. For such modified prices, we deal with an ordinary ski-rental problem, even if the game end is chosen by the adversary and not stochastically, i.e. the ratio is $\mathcal{O}(1)$. Thus, for non-modified prices, the ratio is at most $\mathcal{O}(B/A)$. ◻

Lemma 4. *Let $A < B$ be the thresholds used by PROT_λ. On a (A, B)-low input with parameter $k < s$, for $\lambda \geq s^{4/5}$ and $\gamma \geq 1$, it holds that $\mathcal{R}_{\text{PROT}}(\lambda, \gamma) = \mathcal{O}\left(B/A + A + s \cdot A/\gamma\right)$.*

Proof. If $\gamma < k$, then as in the previous case the competitive ratio is at most $\mathcal{O}(B/A)$.

If $\gamma \geq k$, then $C_{\text{PROT}} = \mathcal{F}(k) + s \cdot p_k \leq \mathcal{F}(k) + s \cdot A \leq k \cdot B + s \cdot A$. During the period $[1, k)$, prices are above A. This means that one of the following cases occurs.

1. $\textsc{Opt}$ buys skis in period $[1, k)$, paying at least $s \cdot A$. In this case, the competitive ratio is

$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} = \mathcal{O}\left(\frac{k \cdot B + s \cdot A}{s \cdot A}\right) = \mathcal{O}(B/A) \ . \tag{5}$$

2. $\textsc{Opt}$ rents skis in period $[1, k)$, paying $\mathcal{F}(k)$. In the remaining period $[k, \gamma]$, it either keeps renting or buys skis, i.e. pays at least $\min\{\gamma - k, s\}$. In this case, the competitive ratio is

$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} \leq \frac{\mathcal{F}(k) + s \cdot A}{\mathcal{F}(k) + \min\{s, \gamma - k\}} \ .$$

We consider two subcases.
(a) If $k \leq \gamma < 2 \cdot k$, then

$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} \leq \frac{\mathcal{F}(k) + s \cdot A}{\mathcal{F}(k)} \leq 1 + \frac{s \cdot A}{k \cdot A} = 1 + s/k = \mathcal{O}(1 + s/\gamma) \ . \tag{6}$$

(b) If $\gamma \geq 2 \cdot k$, then

$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} \leq \frac{s \cdot A}{\min\{s, \gamma - k\}} \leq A + \frac{s \cdot A}{\gamma - k} = \mathcal{O}(A + s \cdot A/\gamma) \ . \tag{7}$$

By combining (5), (6), and (7), we get the lemma. □

Lemma 5. *Let $\lambda \geq s^{4/5}$ and $A < B$ be the thresholds used by* $\textsc{Prot}$*. On a (A, B)-high input with parameter $k < s$, for $\gamma \geq 1$, it holds that $\mathcal{R}_{\text{PROT}}(\lambda, \gamma) = \mathcal{O}\left(B/A + s/B + s/\gamma\right)$.*

Proof. Again if $\gamma < k$, then the competitive ratio is at most $\mathcal{O}(B/A)$, thus we concentrate on the case $\gamma \geq k$. We obtain $C_{\text{PROT}} = \mathcal{F}(k) + s \cdot p_k$.

1. If $k \leq B/2$, then $p_k \geq B$ and during period $[0, k+B/2]$ the price remains in range $[p_k - B/2, p_k + B/2]$. We consider a possibly shorter period $[0, \min\{k+B/2, \gamma\}]$, which actually appears in the game. $\textsc{Opt}$ either rents skis in this period paying at least $\min\{k + B/2, \gamma\} \cdot (p_k - B/2) = \Omega(\min\{B, \gamma\} \cdot p_k)$. or buys skis in this period paying at least $s \cdot (p_k - B/2) = \Omega(s \cdot p_k)$. Thus, the competitive ratio in this case is

$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} = \mathcal{O}\left(\frac{s \cdot p_k}{\min\{B, \gamma, s\} \cdot p_k}\right) = \mathcal{O}\left(1 + \frac{s}{\gamma} + \frac{s}{B}\right) \ . \tag{8}$$

2. If $k > B/2$, then $p_k = B$. We consider two cases.
(a) $\textsc{Opt}$ buys skis in the period $[1, k)$. Then $C_{\text{OPT}} \geq s \cdot A$, $C_{\text{PROT}} = \mathcal{O}(k \cdot B + s \cdot B)$, and thus

$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} = \mathcal{O}(B/A) \ . \tag{9}$$

(b) $\textsc{Opt}$ rents skis in the period $[1, k)$. Note that during this period the price remains above $B/2$. Thus, $C_{\text{OPT}} \geq \mathcal{F}(k)$ and $C_{\text{OPT}} \geq k \cdot B/2 = \Omega(B^2)$. The competitive ratio is then bounded by

$$\frac{C_{\text{PROT}}}{C_{\text{OPT}}} = \mathcal{O}\left(\frac{\mathcal{F}(k) + s \cdot B}{\mathcal{F}(k) + B^2}\right) = \mathcal{O}(s/B) \ . \tag{10}$$

By combining (8), (9), and (10), we get the result. □

Theorem 5. *For $y \in [4/5, 1)$, it holds that $\mathcal{R}_{\text{PROT}}(s^y, \gamma) = \mathcal{O}(1 + s^y/\gamma) \cdot s^{(2-y)/3}$. For $y \geq 1$, it holds that $\mathcal{R}_{\text{PROT}}(s^y, \gamma) = \mathcal{O}(1 + s/\gamma) \cdot s^{1/3}$.*

Proof. Fix any $y \geq 4/5$ and let $\lambda = s^y$. We compute the competitive ratio of $\textsc{Prot}_\lambda$, considering three cases.

1. The input is (A, B)-middle.
   By Lemma 3, the competitive ratio is at most $\mathcal{O}(B/A)$.
2. The input is (A, B)-low with parameter $k < s$.
   By Lemma 4, the competitive ratio is at most $\mathcal{O}(B/A + A + s \cdot A/\gamma)$.
3. The input is (A, B)-high with parameter $k < s$.
   By Lemma 5, the competitive ratio is at most $\mathcal{O}(B/A + s/B + s/\gamma)$.

In total, the competitive ratio is $\mathcal{R} = \mathcal{O}(B/A + A + s/B + s \cdot A/\gamma)$. For $y \in [4/5, 1)$, $A = s^{(2y-1)/3}$ and $B = s^{(y+1)/3}$. In effect, $\mathcal{R}_{\text{PROT}}(s^y, \gamma) = \mathcal{O}(s^{(2-y)/3} + s^{(2y+2)/3}/\gamma) = \mathcal{O}(s^{(2-y)/3} \cdot (1 + s^y/\gamma))$. For $y \geq 1$, $A = s^{1/3}$, $B = s^{2/3}$ and $\mathcal{R}_{\text{PROT}}(s^y, \gamma) = \mathcal{O}(s^{1/3} + s^{4/3}/\gamma) = \mathcal{O}(s^{1/3} \cdot (1 + s/\gamma))$. □

4 Known or Stochastic Game End

In this section, we analyze the performance of the algorithm Prot in scenarios B and C. The former follows immediately, by the bounds of the previous section.

Theorem 6. *The competitive ratio of* PROT_γ *run on a sequence of length* $\gamma = s^y$ *is* $\mathcal{O}(\mathcal{L}(y))$

Proof. Fix any γ and let $y = \log_s \gamma$. The ratio is given by $\mathcal{R}_{\text{PROT}}(\gamma, \gamma)$. We consider several cases.

1. If $y < 0$, then the theorem follows by Lemma 1.
2. If $0 \leq y < 2/3$, then the theorem follows by Theorem 3.
3. If $2/3 \leq y < 4/5$, then by Theorem 4, it holds that $\mathcal{R}_{\text{PROT}}(\gamma, \gamma) = \mathcal{O}(s^{y/2} + s^{(3/2)y}/\gamma + \gamma/s + \gamma^2/s^{(3/2)y}) = \mathcal{O}(s^{y/2}) = \mathcal{O}(\mathcal{L}(y))$.
4. If $y \geq 4/5$, then the theorem by Theorem 5.

□

For Scenario C, we analyze the performance of PROT_λ run on a sequence, whose length is an exponentially distributed random variable $\gamma \sim \text{Exp}(\lambda)$, First, we enumerate several properties of such random variables.

Observation 1 *Let*

$$f(x) = \begin{cases} \frac{1}{\lambda} \cdot e^{-x/\lambda} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

be the probability density function of the exponential distribution with parameter λ. *Then the following relations hold.*

1. $\int_0^\infty x \cdot f(x)\, dx = \lambda$
2. $\int_0^\infty x^2 \cdot f(x)\, dx = \mathcal{O}(\lambda^2)$
3. $\int_1^\infty \frac{1}{x} \cdot f(x)\, dx = \mathcal{O}(\frac{\log \lambda}{\lambda})$

Proof. Let X be an exponentially distributed (with parameter λ) random variable. Then $\mathbf{E}[X] = \lambda$ and variance of X, $\text{Var}(X) = \lambda^2$. By the definition $\int_{i=0}^\infty x \cdot f(x)\, dx = \mathbf{E}[X]$, which proves the first property. For proving the second property, we note that $\int_{x=0}^\infty x^2 \cdot f(x)\, dx = \mathbf{E}[X^2] = (\mathbf{E}[X])^2 + \text{Var}(X) = 2 \cdot \lambda^2$.

For the third property, we observe that

$$\begin{aligned}
\int_1^\infty \frac{1}{x} \cdot f(x)\, dx &= \frac{1}{\lambda} \sum_{j=0}^{\infty} \left(\int_{j\cdot\lambda+1}^{(j+1)\cdot\lambda+1} \frac{1}{x} \cdot \frac{1}{\mathrm{e}^{x/\lambda}}\, dx \right) \\
&\le \frac{1}{\lambda} \sum_{j=0}^{\infty} \frac{1}{e^j} \left(\int_{j\cdot\lambda+1}^{(j+1)\cdot\lambda+1} \frac{1}{x}\, dx \right) \\
&\le \frac{1}{\lambda} \sum_{j=0}^{\infty} \frac{1}{e^j} \cdot \ln\left(\frac{(j+1)\cdot\lambda+1}{j\cdot\lambda+1} \right) \\
&\le \frac{1}{\lambda} \sum_{j=0}^{\infty} \frac{1}{e^j} \cdot \ln\lambda \\
&= \mathcal{O}(1) \cdot \frac{\log\lambda}{\lambda} \ .
\end{aligned}$$

□

Theorem 7. *For any $\lambda = s^y$, the expected value of competitive ratio of* $\textsc{Prot}_\lambda$ *run on a sequence of length $\gamma \sim \mathrm{Exp}(\lambda)$ is $\mathcal{O}(\mathcal{L}(y) \cdot \log\min\{\lambda, s\})$*

Proof. In this proof, we upper-bound the value of

$$\mathbf{E}_\gamma[\mathcal{R}_{\mathrm{PROT}}(\lambda,\gamma)] = \int_0^\infty \mathcal{R}_{\mathrm{PROT}}(\lambda, x) \cdot f(x)\, dx \ ,$$

where $f(x)$ is the density function of exponential distribution (see Observation 1). First, we observe that by Lemma 1, it follows that

$$\begin{aligned}
\mathbf{E}_\gamma[\mathcal{R}_{\mathrm{PROT}}(\lambda,\gamma)] &= \int_0^1 \mathcal{R}_{\mathrm{PROT}}(\lambda, x) \cdot f(x)\, dx + \int_1^\infty \mathcal{R}_{\mathrm{PROT}}(\lambda, x) \cdot f(x)\, dx \\
&= \mathcal{O}(1) + \int_1^\infty \mathcal{R}_{\mathrm{PROT}}(\lambda, x) \cdot f(x)\, dx \ .
\end{aligned} \tag{11}$$

Recall, that by Theorems 3, 4, and 5,

$$\mathcal{R}_{\mathrm{PROT}}(s^y, x) = \mathcal{O}(1) \cdot \begin{cases} 1 + x^2/s & \text{if } y < 2/3 \\ s^{y/2} + s^{(3/2)y}/x + x/s + x^2/s^{(3/2)y} & \text{if } 2/3 \le y < 4/5 \\ (1 + s^y/x) \cdot s^{(2-y)/3} & \text{if } 4/5 \le y < 1 \\ (1 + s/x) \cdot s^{1/3} & \text{if } y \ge 1 \end{cases} \ .$$

By Observation 1, we get that

$$\int_1^\infty \mathcal{R}_{\mathrm{PROT}}(\lambda, x) \cdot f(x)\, dx = \mathcal{O}(\mathcal{R}_{\mathrm{PROT}}(\lambda,\lambda) \cdot \log\lambda) \ . \tag{12}$$

For $\lambda > s$, we may improve this bound as follows.

$$\begin{aligned}
\int_1^\infty \mathcal{R}_{\mathrm{PROT}}(\lambda, x) \cdot f(x)\, dx &= \mathcal{O}(s^{1/3}) \cdot \int_1^\infty (1 + s/x) \cdot f(x)\, dx \\
&= \mathcal{O}(s^{1/3}) \cdot \left(1 + \frac{s}{\lambda} \cdot \log\lambda\right) \\
&= \mathcal{O}(s^{1/3} \cdot \log s) \ .
\end{aligned} \tag{13}$$

Thus, by (11), (12) and (13), we get

$$\mathbf{E}_\gamma[\mathcal{R}_{\mathrm{PROT}}(\lambda,\gamma)] = \mathcal{O}(\mathcal{R}_{\mathrm{PROT}}(\lambda,\lambda) \cdot \log\min\{\lambda, s\}) = \mathcal{O}(\mathcal{L}(y) \cdot \log\min\{\lambda, s\}) \ ,$$

where the second relation follows by Theorem 6. □

5 Lower Bounds for Known Game End Scenario

In this section, we show that PROT_γ achieves asymptotically optimal competitive ratio on the sequences of length γ. Moreover, we show show that this lower bound holds even for randomized algorithms against an oblivious adversary. To illustrate this approach, we first start with an easy case of $\gamma \le s^{2/3}$.

Theorem 8. *Fix $\gamma = s^y$, where $y < 2/3$. No randomized algorithm can achieve a competitive ratio better than $\Omega(\mathcal{L}(y))$.*

Proof. For $y < 1/2$, the lower bound of $\Omega(1)$ follows trivially, thus we assume that $y \in [1/2, 2/3)$.

We call an input curve α-peaky if the price starts at 1, increases till it achieves level α, then drops again to level 1 and remains there. For large α, the dropping part may not occur or be shortened.

As mentioned above, we construct the following probability distribution π over input curves. With probability 1/3 the input presented to the algorithm is $s^{y/2}$-peaky, with probability 1/3 it is $s^{1/2}$-peaky, and with probability 1/3 it is s^y-peaky.

Now we want to analyze the best deterministic algorithm DET and its performance against an input chosen randomly with probability π. Let $C_{\text{DET}}(\alpha)$ and $C_{\text{OPT}}(\alpha)$ be the costs of DET and OPT, respectively, on an α-peaky input. By the Yao min-max principle, it follows that the competitive ratio of any randomized algorithm against an oblivious adversary is at least

$$\mathcal{R} \ge \mathbf{E}_\pi\left[\frac{C_{\text{DET}}}{C_{\text{OPT}}}\right] = \frac{1}{3}\cdot\frac{C_{\text{DET}}(s^{y/2})}{C_{\text{OPT}}(s^{y/2})} + \frac{1}{3}\cdot\frac{C_{\text{DET}}(s^{1/2})}{C_{\text{OPT}}(s^{1/2})} + \frac{1}{3}\cdot\frac{C_{\text{DET}}(s^{y})}{C_{\text{OPT}}(s^{y})} \ . \tag{14}$$

DET has essentially two choices. It may opt to never buys skis or it may choose a parameter r and buy if the price achieves r. If the price starts to drop before it hits level r, it does not make sense for DET to buy skis in the remaining period, as this cost would be greater then renting skis till time $\gamma < s^{2/3}$.

Thus, we consider a few cases

1. DET chooses $r \in [s^0, s^{y/2})$. On a $s^{y/2}$-peaky input, an algorithm renting the skis for the whole game, pays at most $\mathcal{O}((s^{y/2})^2 + s^y) = \mathcal{O}(s^y)$, and thus $C_{\text{OPT}}(s^{y/2}) = \mathcal{O}(s^y)$. As all terms occurring in (14) are non-negative, we may omit some of the summands occurring there. Therefore,
$$\mathcal{R} \ge \frac{1}{3}\cdot\frac{C_{\text{DET}}(s^{1/2})}{C_{\text{OPT}}(s^{1/2})} \ge \frac{1}{3}\cdot\frac{s\cdot r}{s^y} = \Omega(s^{1/3}) = \Omega(s^{2y-1}) \ . \tag{15}$$
2. DET chooses $r \in [s^{y/2}, s^{1/2})$. Note that $C_{\text{OPT}} \le s$, as OPT may buy skis at the very beginning.
$$\mathcal{R} \ge \frac{1}{3}\cdot\frac{C_{\text{DET}}(s^{1/2})}{C_{\text{OPT}}(s^{1/2})} \ge \frac{1}{3}\cdot\frac{s\cdot r}{s} = \Omega(s^{y/2}) = \Omega(s^{2y-1}) \ . \tag{16}$$
3. DET chooses $r \ge s^{1/2}$ or rents skis for the whole game. On a s^y-peaky input these strategies have costs $\Omega(s \cdot r) = \Omega(s^{2y})$ and $\Omega(s^{2y})$, respectively. Again $C_{\text{OPT}} \le s$. Thus,
$$\mathcal{R} \ge \frac{1}{3}\cdot\frac{C_{\text{DET}}(s^{y})}{C_{\text{OPT}}(s^{y})} = \Omega(s^{2y-1}) \ . \tag{17}$$

By (15), (16), and (17), it follows that the bound holds for any strategy of DET. □

For the lower bound for $\gamma \ge s^{2/3}$, we first prove it for a deterministic algorithm. Although —as we prove a stronger result in the next theorem— this theorem is redundant, it illustrates key concepts which are used later.

Theorem 9. *Fix $\gamma = s^y$, where $y \ge 2/3$ and a deterministic algorithm* DET. *The competitive ratio of* DET *is at least $\Omega(\mathcal{L}(y))$.*

Proof. Consider the following adaptively generated input curve. In the first step, the price is equal to s^a. Then the price increases by 1 each day. When the algorithm buys skis, the price begins to fall, till it achieves 1 and it remains at this level to the end of the game.

Let $f = \min\{y, 1\}$. Let $a \in [0, f/2]$; we will specify the exact value of a later. We apply the generic approach above with value a. We consider several possible strategies of DET; we compare the cost of DET to the cost of an offline strategy OFF. Obviously, $C_{\text{OPT}} \leq C_{\text{OFF}}$.

1. DET buys skis for $x \in [s^a, s^{f/2})$. Then $C_{\text{DET}} \geq s^{1+a}$. OFF rents till moment s and then buys skis (for price 1). OFF pays $\mathcal{O}(x^2 + s^f)$ for renting and s for buying skis. OFF buys skis only if $f = 1$. Therefore, $C_{\text{OFF}} = \mathcal{O}(x^2 + s^f) = \mathcal{O}(s^f)$ and $C_{\text{DET}}/C_{\text{OFF}} = \Omega(s^{1-f+a})$.
2. DET buys skis for $x \in [s^{f/2}, s^{(1+a)/2})$. Then $C_{\text{DET}} \geq s \cdot x$, whereas OFF employs the strategy from the previous case, paying $\mathcal{O}(x^2 + s^f) = \mathcal{O}(x^2)$. Thus, $C_{\text{DET}}/C_{\text{OFF}} = \Omega(s/x) = \Omega(s^{1-(1+a)/2}) = \Omega(s^{(1-a)/2})$.
3. DET buys skis for $x \geq s^{(1+a)/2}$, paying $C_{\text{DET}} \geq s \cdot s^{(1+a)/2}$. OFF buys skis at the beginning paying $s \cdot s^a$. Therefore, $C_{\text{DET}}/C_{\text{OFF}} = \Omega(s^{(1+a)/2-a}) = \Omega(s^{(1-a)/2})$.
4. DET always rents, paying at least $\Omega(s^{2f})$. OFF buys skis at the beginning, paying $s \cdot s^a$. Thus, $C_{\text{DET}}/C_{\text{OFF}} = \Omega(s^{2f-1-a})$.

As DET may freely choose on of the strategies above, we obtain that for any fixed y, the competitive ratio of DET is

$$\mathcal{R}_{\text{DET}} = \Omega\left(\max_a \left(\min\left\{s^{1-f+a}, s^{(1-a)/2}, s^{2f-1-a}\right\}\right)\right) \quad .$$

1. For $y \in [2/3, 4/5)$, we choose $a = (3/2)y - 1$. In this case $\mathcal{R}_{\text{DET}} = \Omega(s^{y/2})$.
2. For $y \in [4/5, 1)$, we choose $a = (2y-1)/3$. In this case $\mathcal{R}_{\text{DET}} = \Omega(s^{(2-y)/3})$.
3. For $y \geq 1$, we choose $a = 1/3$. In this case $\mathcal{R}_{\text{DET}} = \Omega(s^{1/3})$.

These cases together conclude the proof. □

Theorem 10. *Fix $\gamma = s^y$, where $y \geq 2/3$. No randomized algorithm can achieve a competitive ratio better than $\Omega(\mathcal{L}(y))$.*

Proof. As in the proof of Theorem 8, we use Yao min-max principle. We use similar construction to the one occurring in the proof of Theorem 9, but this time the adversary is not allowed to base its decision on the behavior of the algorithm. Let $f = \min\{y, 1\}$. Let $a \in [0, f/2]$; we will specify the exact value of a later.

We modify the definition of an α-peaky input: now such a sequence starts not from level 1 but from s^a; other details remain intact.

We construct the following probability distribution π over input curves and present it to the best deterministic algorithm DET. With probability 1/3 the input presented to the algorithm is $s^{f/2}$-peaky and with probability 1/3 it is s^f-peaky. The remaining probability 1/3 is distributed uniformly over α-peaky inputs for $\alpha \in [s^{f/2}, 2 \cdot s^{(1+a)/2}]$. This means that the probability density function is

$$f(\alpha) = \begin{cases} \frac{1}{3} \cdot \left(2 \cdot s^{(1+a)/2} - s^{f/2}\right)^{-1} & \text{if } \alpha \in [s^{f/2}, 2 \cdot s^{(1+a)/2}] \\ 0 & \text{otherwise} \end{cases}$$

Note that for $\alpha \in [s^{f/2}, 2 \cdot s^{(1+a)/2}]$, $f(\alpha) = \Theta(s^{-(1+a)/2}))$.

By the Yao min-max principle, it follows that the competitive ratio of any randomized algorithm against an oblivious adversary is at least

$$\mathcal{R} \geq \mathbf{E}_\pi \left[\frac{C_{\text{DET}}}{C_{\text{OPT}}}\right] = \frac{1}{3} \cdot \frac{C_{\text{DET}}(s^{f/2})}{C_{\text{OPT}}(s^{f/2})} + \frac{1}{3} \cdot \frac{C_{\text{DET}}(s^f)}{C_{\text{OPT}}(s^f)} + \int_{s^{f/2}}^{2s^{(1+a)/2}} \frac{C_{\text{DET}}(\alpha)}{C_{\text{OPT}}(\alpha)} f(\alpha)\, d\alpha \quad .$$

Again, DET has two choices. It may rent forever or it may choose a parameter $r \in [s^a, s^f]$ and buy if the price achieves r. We analyze this behavior in four cases below.

1. DET chooses $r \in [s^a, s^{f/2})$. With probability 1/3, input is $s^{f/2}$-peaky. For such an input, $C_{\text{DET}} \geq s^{1+a}$ and $C_{\text{OPT}} = \mathcal{O}((s^{f/2})^2 + s^f) = \mathcal{O}(s^f)$. Thus,

$$\mathcal{R} \geq \frac{1}{3} \cdot \frac{C_{\text{DET}}(s^{f/2})}{C_{\text{OPT}}(s^{f/2})} = \Omega\left(s^{1-f+a}\right) \ . \tag{18}$$

2. DET chooses $r \in [s^{f/2}, s^{(1+a)/2})$. Note that $C_{\text{OPT}}(\alpha) = \mathcal{O}(\alpha^2 + s^f) = \mathcal{O}(\alpha^2)$. Then

$$\begin{aligned} \mathcal{R} \ &\geq \int_r^{2s^{(1+a)/2}} \frac{C_{\text{DET}}(\alpha)}{C_{\text{OPT}}(\alpha)} f(\alpha)\, d\alpha \\ &= \Theta(s^{-(1+a)/2})) \cdot \int_r^{2s^{(1+a)/2}} \frac{s \cdot r}{\alpha^2}\, d\alpha \\ &= \Theta(s^{-(1+a)/2})) \cdot \left(\frac{s \cdot r}{r} - \frac{s \cdot r}{2s^{(1+a)/2}}\right) \\ &= \Theta(s^{-(1+a)/2})) \cdot (s - s/2) \\ &= \Theta(s^{(1-a)/2}) \ . \end{aligned} \tag{19}$$

3. DET chooses $r \in [s^{(1+a)/2}, s^f]$. With probability 1/3, input is s^f-peaky. For such an input, $C_{\text{DET}} \geq s \cdot s^{(1+a)/2}$. On the other hand, OPT may buy skis at the beginning paying $C_{\text{OPT}} = s \cdot s^a$. In this case

$$\mathcal{R} \geq \frac{1}{3} \cdot \frac{C_{\text{DET}}(s^f)}{C_{\text{OPT}}(s^f)} = \Omega\left(s^{(1-a)/2}\right) \ . \tag{20}$$

4. DET never buys skis. Then, as in the analysis from the previous case, we note that with probability 1/3 input is s^f-peaky, and $C_{\text{DET}} \geq s^{2f}$. Then

$$\mathcal{R} \geq \frac{1}{3} \cdot \frac{C_{\text{DET}}(s^f)}{C_{\text{OPT}}(s^f)} = \Omega\left(s^{2f-1-a}\right) \ . \tag{21}$$

As DET may freely choose on of the strategies above, by (18), (19), (20), and (21), we obtain that for any fixed y, the competitive ratio of DET is

$$\mathcal{R}_{\text{DET}} = \Omega\left(\max_a \left(\min\left\{s^{1-f+a}, s^{(1-a)/2}, s^{2f-1-a}\right\}\right)\right) \ .$$

This means that the lower bound is asymptotically the same as in the deterministic case (see the proof of Theorem 9). □

6 Solution for the discrete model

Finally, we show that our algorithms can be made to work in the discrete model, without increasing their competitive ratio. Although the same theorem applies for all algorithms, for simplicity of the proof we use the following feature of our algorithms: they do not require that the price curve is continuous, but only that it satisfies the Lipschitz condition.

Theorem 11. *For any algorithm* CONT*, which is* $\mathcal{R}$*-competitive in the continuous model (provided that the price curve satisfies the Lipschitz condition), there is an algorithm* DISC*, which is* $\mathcal{R}$*-competitive in the discrete model.*

Proof. We fix any discrete infinite sequence of prices $I = p_1, p_2, p_3, \ldots$ and we show an algorithm DISC which performs well on this sequence. First, we create a curve of prices $I' = p'_t$, such that $p'_t = p_{\lceil t \rceil}$. Note that I' can be created in online fashion. I' is not continuous, but it satisfies Lipschitz condition. DISC runs CONT on I' and if CONT buys skis at time t', then DISC buys skis (for the same price) at day $t = \lfloor t' \rfloor + 1$.

First, we observe that $C_{\text{DISC}}(I) \leq C_{\text{CONT}}(I')$. This relation is trivial if Cont and Disc always rent. If Cont buys at time t', Disc buys at day $t = \lfloor t' \rfloor + 1$, and we get $C_{\text{CONT}}(I') = \mathcal{F}(t') + s \cdot p'_{t'} \geq \sum_{i=1}^{\lfloor t' \rfloor} p_i + s \cdot p_t = C_{\text{DISC}}(I)$.

Second, we prove that $C_{\text{OPT}}(I') \leq C_{\text{OPT}}(I)$. We consider the following algorithm Off for I', which buys at time $t-1$ if Opt on I buys at day t. Clearly, $C_{\text{OPT}}(I') \leq C_{\text{OFF}}(I') = C_{\text{OPT}}(I)$.

Since by the theorem assumption, $C_{\text{CONT}}(I') \leq \mathcal{R} \cdot C_{\text{OPT}}(I')$, we get the following relation

$$\text{Disc}(I) \leq \text{Cont}(I') \leq \mathcal{R} \cdot C_{\text{OPT}}(I') \leq \mathcal{R} \cdot C_{\text{OPT}}(I) \ ,$$

which implies the theorem. □

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